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APQWs IN COMPLEX PLANE: APPLICATION TO FREDHOLM INTEGRAL EQUATIONS

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Abstract. We consider a class of orthonormal quasi-wavelets and introduce quasi-wavelet method for solving Fredholm integral equations of the second kind in complex plane. We analyze the convergence of the anti-periodic quasi-wavelet method for solving linear Fredholm integral equation. The high accuracy and the wide applicability of APQWs approach is demonstrated with numerical example.

Keywords: Fredholm integral equation; Anti-periodic quasi-wavelet; Collocation method; Complex plane.

AMS Subject Classification: 47A56, 45B05, 47H10, 42C40.

1. Introduction

There are some problems in physics [4] which lead to the integral equation of the second kind

$$u(x)=f(x)+\int_0^{2\pi} k(x,t)u(t)dt, \qquad 0\le x\le 2\pi \tag{1}$$

where k and f are continuous complex anti-periodic functions and u is an unknown complex function to be found.

While several numerical methods for approximating the solution of real integral equations with high accuracy are known [1,7,6], for complex one only a few methods are considered [2].

Wavelet analysis is a new branch of mathematics and has made a great impact on science and engineering in the past decade [5]. For many unsteady and localized problems, wavelet analysis is superior to Fourier analysis, due to its time-frequency localization. As a result, wavelet analysis is widely applied to many scientific and engineering fields, including signal processing, image manipulation, and solution of differential equations [9]. Based on the multi-resolution analysis [9,10], Quasi-wavelets have much better localized property than their original wavelets.

In this paper we describe a numerical scheme using anti-periodic quasi wavelets (APQWs) for solving Eq. (1). The theory of APQWs based on B-spline functions is found in [3, 4].

The main characteristic of our method is to convert integral equation (1) into an algebraic system by expanding the unknown function in terms of APQWs with unknown coefficients.

2. Preliminary

By Denoting $\overset{\circ}{L}_2[0,T] := \left\{ f \;\middle|\; \int_0^T |f(x)|^2 dx \prec \infty, f(x) = f(x+T) \right\}$

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The inner product of two functions f and g in $\overset{\circ}{L}_2[0,T]$ is defined by

$$\langle f, g \rangle := \frac{1}{T} \int_0^T f(x) \overline{g(x)} dx \tag{3}$$

Definition 1. The periodic B-spline function is defined by

$$B_i^{n,m}(x) := B_i^n(x, h_m) = (-1)^{n+1} \left(y_{i+n+1}^m - y_i^m \right) \left[y_i^m, \ldots, y_{i+n+1}^m \right]_y (x-y)_+^n \tag{4}$$

And the space of periodic splines is spanned by the B-splines, that is,

$$V_m := span \left\{ \overline{B}_i^{n,m}(x) = B_i^{n,m}(x) + B_{i+K(m)}^{n,m}(x),\ x \in [0, 2\pi],\ K(m) = 3^m k \right\} \tag{5}$$

$\left\{ \overline{B}_i^{n,m}(x) \right\}_{i=0}^{K(m)-1}$ is a basis of V_m, but it is not orthogonal. We construct an orthogonal basis for V_m. For this purpose, we define

Definition 2.

$$A_{k,3}^{n,m}(x) := C_k^{n,m} (k(m))^{n+1} \sum_{v \in \mathbf{Z}} \left(\frac{\sin\left(v + \frac{k}{k(m)} \right)\pi}{(vk(m) + k)\pi} \right)^{n+1} \exp\left(i \left(k + vk(m) \right) x \right)$$
$$C_v^{n,m} = \left[t_0 + 2 \sum_{\lambda=1}^{n} B_0^{2n+1,m}(\lambda, 1) \cos\left(\lambda v h_j \right) \right]^{-\frac{1}{2}} \tag{6}$$

Lemma 1. The set of functions $\left\{ A_{v,3}^{n,m}(x) \right\}_{v=0}^{K(m)-1}$ is an orthonormal basis for V_m.

We have the following 3-scale equations

$$A_{v,3}^{n,m}(x) = a_{v,v} A_{v,3}^{n,m+1}(x) + a_{v,v+k(m)} A_{v+k(m),3}^{n,m+1}(x) + a_{v,v+2k(m)} A_{v+2k(m),3}^{n,m+1}(x)$$
$$a_{v,v+\lambda K(m)} = < A_{v,3}^{n,m}, A_{v+\lambda k(m),3}^{n,m+1} > \quad for \quad \lambda = 0,1,2 \tag{7}$$

Proof: See [2]

Definition 3.

$$D_{j,3}^{n,m}(x)=\begin{cases} -\bar{a}_{j,j+k(m)}a_{j,j}q_j\, A_{j,3}^{n,m+1}(x)+\dfrac{1}{q_j}\, A_{j+k(m),3}^{n,m+1}(x) \\ \qquad -\bar{a}_{j,j+k(m)}a_{j,j+2k(m)}q_j\, A_{j+2k(m),3}^{n,m+1}(x) & 0\le j\le k(m)-1 \\ -\bar{a}_{j-k(m),j+k(m)}q_{j-k(m)}A_{j-k(m),3}^{n,m+1}(x) \\ \qquad +\bar{a}_{j-k(m),j-k(m)}q_{j-k(m)}A_{k+k(m),3}^{n,m+1}(x) & k(m)\le j\le 2k(m)-1 \end{cases} \tag{8}$$

Where

$$a_{i,j}=< A_{i,3}^{n,m}, A_{j,3}^{n,m+1}>, \qquad q_k=\left[1/\left(1-\left|a_{k,k+k(m)}\right|^2\right)\right]^{\frac{1}{2}} \tag{9}$$

Lemma 2. $\left\{D_{j,3}^{n,m}(x)\right\}_{j=0}^{2k(m)-1}$ is an orthonormal basis for W_m , and $V_{m+1}=V_m\oplus W_m$, where W_m is the orthogonal complementary space of V_m in V_{m+1}.
Proof: See [3].

3. Anti-periodic quasi-wavelets

In this section, we review the APQWs in the framework of the recursive periodic quasi- wavelets construction given in [4] based. A function $f(x)$ is called anti-periodic π if $f(x+\pi)=-f(x)$ and the space of anti-periodic spline functions is defined

$$V_m^n:=\left\{ f \left| \begin{array}{l} 1)\ f(x)\in\pi_n \quad if \quad x\in\left[y_v^m, y_{v+1}^m\right) \\ 2) f\in c^{(n-1)}([0,\pi]) \\ 3) f^{(j)}(x+\pi)=-f^{(j)}(x) \quad j=0,1,\dots n-1 \end{array} \right. \right\} \tag{10}$$

Where $y_j^m=y_0^m+jh_m,\ j\in\mathbf{Z},\ y_0^m=-n_0h_m,\ n_0=\dfrac{n+1}{2},\ h_m=\dfrac{h}{3^m},\ h=\dfrac{\pi}{L},\ L\in\mathbf{Z}_+$

Definition 4.

$\overset{\circ}{E}{}_i^{n,m}(x):=\overset{\circ}{B}{}_i^{n,m}(x)-\overset{\circ}{B}{}_{i+l(m)}^{n,m}(x)$, where $\overset{\circ}{B}{}_{i+l(m)}^{n,m}$ is the periodic extension of $\overline{B}_{i+l(m)}^{n,m}$

Lemma 3. $\left\{E_i^{n,m}\right\}_{i=0}^{L(m)-1}$ is a basis of the space V_m^a , where $E_i^{n,m}(x)$ is the restriction of $\overset{\circ}{E}{}_i^{n,m}(x)$ on $[0,2\pi]$.

Proof: See [3].

Definition 5.

$$A_{n,m}^{a,j}(x) := \frac{1}{2\pi}\left[A_{2j-1,3}^{n,m}(x) - A_{2j-1,3}^{n,m}(x-\pi)\right] \tag{11}$$

Lemma 4.

$\left\{A_{n,m}^{a,j}(x)\right\}_{j=0}^{L(m)-1}$ is an is an orthonormal basis in V_m^a.

Proof: See [4].

Definition 6.

$$D_{n,m}^{a,j}(x) := \frac{1}{2\pi}\left[D_{2j-1,3}^{n,m}(x) - D_{2j-1,3}^{n,m}(x-\pi)\right] \tag{12}$$

Lemma 5.

$\left\{D_{n,m}^{a,j}(x)\right\}_{j=0}^{2L(m)-1}$ is an orthonormal basis in W_m^a where W_m^a is complementary subspace V_m^a in V_{m+1}^a

Proof: See [4].

We call $A_{n,m}^{a,j}$ the father anti-periodic quasi-wavelet and $D_{n,m}^{a,j}$ the mother anti-periodic quasi-wavelet.

4. Solution of Fredholm integral equations of the second kind

In this section, the anti-periodic quasi-wavelets provided in previous section are used to solve integral equation (1). The classic collocation method for Fredholm integral equations of the second kind given in Eq. (1) consists of seeking and approximating solution of the form

$$u_m = \sum_{j=0}^{L(m)-1} c_j A_{n,m}^{a,j} \tag{13}$$

If substituting, we find that

$$r_n(x) = \int_0^{2\pi} k(x,t))u_n(t)dt + f(x) - u_n(x) \tag{14}$$

Where $r_n(x)$ is the residual such that $r_n(x)=0$ for $u_n(x)=u(x)$. Our goal is to compute $c_0, c_1, ..., c_{L(m)-1}$. We compute c_j such that $r_n(x_j)=0$ where $\{x_j\}$ is the set of distinct collocation node points from $[0,2\pi]$ defined by $x_j = \frac{2\pi}{L(m)-1} j$.

5. Convergence Analysis

We discuss the convergence of the anti-periodic quasi-wavelets method for the linear Fredholm integral equations (1). For this purpose, firstly, we need the following lemmas.

Lemma 6. Assume $X = Y = L^2[0,2\pi]$ and κ be the integral operator associated with kernel $k(x,t)$. Let

$$M = \left[\int_0^{2\pi}\int_0^{2\pi}|k(x,t)|^2 dxdt\right]^{\frac{1}{2}} \tag{15}$$

And assume $M < 1$. Then operator $I - \kappa : x \to x$ has a bounded inverse $(I-\kappa)^{-1}$.

Proof: for $\mathring{L}_2[0,T]$, we use the Cauchy-Schwartz inequality

$$\begin{gathered}\|\kappa u\|^2 = \int_0^{2\pi}\left|\int_0^{2\pi} k(x,t)u(t)dt\right|^2 dx \le \\ \int_0^{2\pi}\left\{\int_0^{2\pi}|k(x,t)|^2 dt\right\}\left\{\int_0^{2\pi}|u(t)|^2 dt\right\}dx = M^2\|u\|_2^2 < \|u\|_2^2\end{gathered} \tag{16}$$

It shows that $\|\kappa\| < 1$ and then $I - \kappa : x \to x$ has a bounded inverse.

Lemma 7.

Let P_m be a bounded projection operator of $\mathring{L}_2[0,2\pi]$ to V_m, for all $f \in \mathring{L}_2[0,2\pi]$,

$$\lim_{m\to\infty}\|f - P_m f\|_2 = 0 \tag{17}$$

Proof: See [4].

Lemma 8. Let X, Y be Banach spaces, and $\kappa_n : X \to Y$ be a sequence of bounded linear operators such that $\{\kappa_n(x)\}$
is convergent, for all $x \in X$, then the convergence is uniform on compact subsets of X.

Proof: See [1].

Lemma 9.

Let $\{P_n\}$ be a family of bounded projections on $\mathring{L}_2[0,2\pi]$ with $P_n(x) \to x$, *for* $x \in \mathring{L}_2[0,2\pi]$ then

$$\|\kappa - P_m\kappa\|_2 \to 0 \qquad as \qquad n \to \infty \tag{18}$$

Proof: See [1].

Now we can show that the inverse of $I - P_n\kappa$ exits.

Lemma 10. Assume $A : X \to X$ is bounded, with X be a Banach space, and assume $I-A:X\to X$ is 1-1 and onto. Further assume $\|A-P_nA\|_2\to 0$, then for all sufficiently large n $n\geq N$, the operator $(I-P_nA)^{-1}$ exists. Moreover it is uniformly bounded: $\sup_{n\geq N}\left\|(I-P_nA)^{-1}\right\|<\infty$

Proof: See [8].

Corollary 1. Since the operator κ satisfies in Lemma 10., therefore $I-P_n\kappa$ has bounded inverse. So, there exists a Constant M_1 independent of n such that

$$\left\|(I-P_m\kappa)^{-1}\right\|_2<M_1$$

Theorem 1. Assume that u_n is an approximate solution and u is the exact solution of Eq. (1), then $\|u-u_n\|_2\to 0$

Proof: We have $(I-P_n\kappa)u_n=P_nf$, then $(I-P_n\kappa)(u-u_n)=u-P_nu$

From Lemma 10. $I-P_n\kappa$ is invertible for large n, so

$$u-u_n=(I-P_n\kappa)^{-1}(u-P_nu)\Rightarrow\|u-u_n\|=\left\|(I-P_n\kappa)^{-1}\right\|\|(u-P_nu)\| \tag{19}$$

By using Lemmas 3 and 6, $\|u-u_n\|_2\to 0$.

6. Numerical example

In this section we use a numerical example to test and show the efficiency and accuracy of the proposed method. The computations associated with the examples were performed using Mathematica 7.

Example 1. Consider the following Complex integral equation

$$u(x)=\cos(x)+(i-\pi)\sin(x)+\int_0^{2\pi}\sin(x)\cos(t)u(t)dt \tag{20}$$

With exact solution $u(x)=\cos(x)+i\sin(x)$ that is a graph of circle in complex plane. We used anti-periodic quasi-wavelet of order $n=2$. The results $m=1,2$ are shown in Fig 1 and Tables 1 and 2.

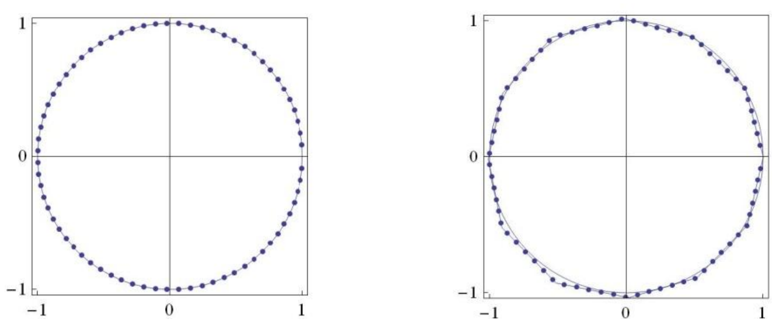

Fig.1. Comparison between approximate and exact solution of example 1 (a) for m=2; (b) for m=1;

(Approximate solution; ----- Exact solution)

Table 1.Numerical results for imaginary part of example 1

x	Absolute errors for m=1	Absolute errors for m=2
1	3.58528×10^{-2}	4.9154×10^{-5}
2	2.15032×10^{-2}	1.8847×10^{-5}
3	4.22097×10^{-3}	4.528×10^{-5}
4	1.01678×10^{-2}	4.07305×10^{-5}
5	3.20566×10^{-3}	2.63795×10^{-6}
6	5.6325×10^{-3}	5.18777×10^{-5}

Table 2.Numerical results for real part of example 1

x	Absolute errors for m=1	Absolute errors for m=2
1	2.12786×10^{-2}	1.32633×10^{-4}
2	3.40295×10^{-2}	1.69907×10^{-5}
3	1.80469×10^{-2}	2.06563×10^{-6}
4	2.9898×10^{-2}	9.89501×10^{-5}
5	2.2688×10^{-2}	7.95982×10^{-5}
6	1.73541×10^{-2}	2.95619×10^{-7}

Example 2. Consider the following Complex integral equation

$$u(x) = i\cos^3(x) + \mathrm{Sin}^3(x) - i\frac{5\pi}{8}\sin(x) + \int_0^{2\pi}\sin(x)\cos^3(t)u(t)dt \qquad (21)$$

With exact solution $u(x) = \cos^3(x) + i\sin^3(x)$. We used anti-periodic quasi-wavelet of order $n = 3$. The results $m = 2$ are shown in Figs 2, 3, 4 and Table 3.

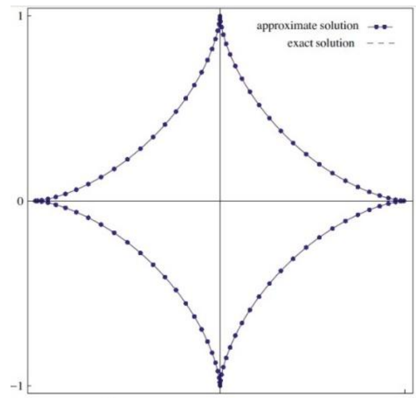

Fig.2. Comparison between approximate and exact solution of example 2;

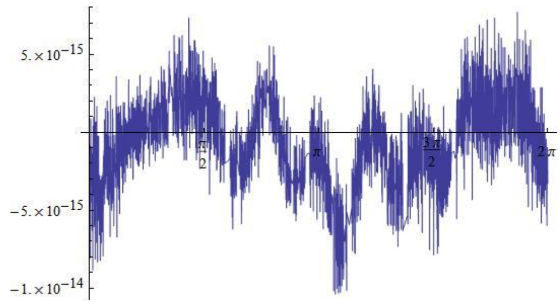

Fig.3. Absolute values of errors for real part of example 2;

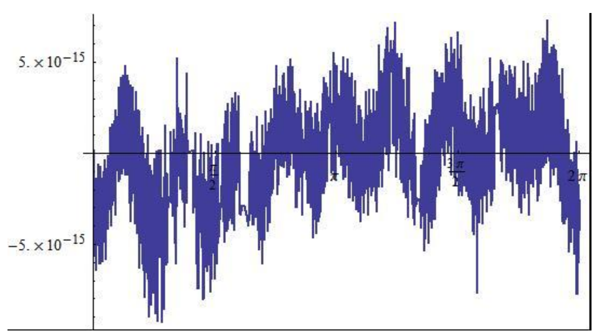

Fig.4. Absolute values of errors for imaginary part of example 2;

Table 3.Numerical results for Absolute values of errors for Example2.

x	Absolute errors for imaginary part	Absolute errors for real part
1	3.2752×10^{-15}	1.2212×10^{-15}
2	$4.6629{\times}10^{-15}$	$2.7756{\times}10^{-15}$
3	$1.9984{\times}10^{-15}$	$7.6848{\times}10^{-16}$
4	$2.7201{\times}10^{-15}$	$2.2205{\times}10^{-16}$
5	$1.131{\times}10^{-15}$	$1.1102{\times}10^{-15}$
6	$2.8866{\times}10^{-15}$	$1.4225{\times}10^{-16}$

References

1. Atkinson. K.E, The Numerical Solution of Integral Equations of the Second Kind, Cambridge University Press.
2. Beiglo H., Gachpazan M., PQWs in complex plane, Application to Fredholm integral equations, Applied Mathematical Modelling, Vol.37, No.22, 2013, pp.9077-9085.
3. Chen H.L., Antiperiodic wavelets, J. Comp. Math., Vol.14, No.1, 1996, pp.32-39.
4. Chen H.L., Complex Harmonic Splines, Periodic quasi-wavelets, Theory and Applications, Kluwer Academic Publishers, 1999.
5. Chui CK., An Introduction to Wavelets [M], Academic Press: San Diego, 1992.
6. Erfanian M., Gachpazan M., Beiglo H., Rationalized Haar wavelet bases to approximate solution of nonlinear Fredholm integral equations with error analysis, Applied Mathematics and Computation, Vol.265, 2015, pp.304–312.
7. Kress R., Linear Integral Equations, Springer, 1989.
8. Schumaker L., Spline Functions: Basic Theory, Wiley, 1981.
9. Mallat S., Hwang WL., Singularity detect and processing with wavelets. IEEE Transactions on Information Theory, Vol.38, No.2, 1992, pp. 617-634.
10. Mallat S., A theory of multiresolution signal decomposition: the wavelet representation. IEEE Transactions on pattern Analysis and Machine Intelligence Vol.11, 1989, pp.69-87.

Kompleks müstəvidə APQW və onun inteqral Fredqolm tənliklərinə tətbiqləri

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XÜLASƏ

Ortonormal kvazi-dalğalar sinfinə baxılır və kompleks müstəvidə ikinci növ Fredholm tənliklərinə tətbiq edilmək üçün kvazi-dalğalar üsulu təklif edilir. inteqral xətti Fredholm tənlikləri üçün olan anti-periodik dalğa üsulunun yığılması tədqiq edilir. APQW üsulunun yüksək dəqiqliyi və tətbiqləri misallar vasitəsilə nümayiş etdrilir.

Açar sözlər: Fredholm inteqral tənliyi, antiperiodik kvazi dalğalar, коllokasiya metodu, kompleks müstəvi.

APQWs в комплексной плоскости: применение к интегральным уравнениям Фредгольма

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РЕЗЮМЕ

Рассмотривается класс ортонормированых квази-вейвлетов и предлагается метод квази-вейвлетов для решения интегральных уравнений Фредгольма второго рода в комплексной плоскости. Мы анализируем сходимость анти-периодического метода квази-вейвлетов для решения линейного интегрального уравнения Фредгольма. Высокая точность и широкая применимость APQWs подхода продемонстрирована численным примером.

Ключевые слова: интегральное уравнение Фредгольма; Анти-периодические квази-вейвлетов; Метод коллокации; Комплексной плоскости.